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We consider the random-bond Potts model in the large-Q limit and calculate the excess entropy, S_Γ, of a contour, Γ, which is given by the mean number of Fortuin-Kasteleyn clusters which are crossed by Γ. In two dimensions S_Γ is proportional to the length of Γ, to which - at the critical point - there are universal logarithmic corrections due to corners. These are calculated by applying techniques of conformal field theory and compared with the results of large scale numerical calculations. The central charge of the model is obtained from the corner contributions to the excess entropy and independently from the finite-size correction of the free-energy as: $\lim_{Q\to\infty} c(Q)/\ln Q = 0.74(2)$, close to previous estimates calculated at finite values of Q.

I. INTRODUCTION

Entropy represents a fundamental concept in different domains of science, such as in information theory[1], in quantum systems[2] and in classical statistical mechanics. In quantum systems the entanglement entropy turned out a very important indicator of new and exotic phases and quantum phase transitions[2–5]. Its analogue in information theory is the mutual information and in classical statistical mechanics the excess entropy. Also in classical systems the excess entropy can be defined as mutual information, where the probabilities are given as Boltzmann weights[6–8]. In these calculations the system is divided in two (or more) parts, and the excess entropy is basically associated with the interface separating the subsystems. Due to this, the excess entropy is proportional to the surface of the interface, which is called the area law. In critical systems, however, usually there are universal corrections to the area law, which are logarithmic in the linear extent of the interface. In conformally invariant systems, such as in one-dimensional quantum and in two-dimensional classical models the central charge of the conformal field theory can be deduced from the size-dependence of the critical entanglement entropy[2–5] and mutual information[8], respectively.

Concerning classical statistical mechanics most of the studies described in the previous paragraph are performed on non-random systems. It is known, however, that systems with quenched disorder are also conformally invariant, provided the properties of averaged quantities (magnetization, correlation function, etc.) are concerned. Therefore it is of interest to study the scaling properties of the excess entropy in random systems, too. For this purpose we consider the two-dimensional Q-state Potts model[9] in the presence of bond disorder[10]. In two dimensions (2D) the phase transition in the random bond Potts model (RBPM) is of second order[11–13] for any value of Q, even in the limit $Q \to \infty$. The critical behavior of this model has been studied by different methods[14–18], in particular the critical exponents and the central charge has been calculated, mainly for $Q > 4$, in which case the phase transition in the non-random model is of first order[19]. Special attention has been paid to the model in the large-Q limit, in which case the critical parameters are found to be smooth function of $1/\ln Q$[20]. For example from the numerical data calculated at large, but finite values of Q the central charge is conjectured to be[20] $\lim_{Q\to\infty} c(Q)/\ln Q \equiv c' = 1/(2\ln 2) = 0.7213$.

Later, it has been shown[21] that the model can be studied directly at the limiting value $Q \to \infty$, when in the random cluster representation[22] the partition function of the model is dominated by one term, the so called *optimal graph*. This means that thermal fluctuations are negligible compared to disorder fluctuations, thus the critical behavior of the system is controlled by a so called infinite disorder fixed point[26]. The optimal graph of the RBPM has been calculated by a combinatorial optimization method[23], which provides the exact value of the partition function for a given sample, i.e. for a given realization of the disorder. From the numerical data exact values of the critical exponents are conjectured[24,25] through an expected relation with the exactly known infinite disorder fixed point of the random transverse-field Ising chain[27]: $x_m = (5-\sqrt{5})/4$ (bulk magnetization), $x_m^s = 0.5$ (surface magnetization) and $\nu = 1$ (correlation length).

In this paper we study the scaling properties of the excess entropy in the RBPM in the large-Q limit. For this we consider a subset of bonds, Γ, and calculate the corresponding excess entropy, S_Γ. If Γ is a closed loop, separating a subsystem, $\mathcal{A}$, from the rest of the system, $\mathcal{B}$, then S_Γ is the mutual entropy $S_\Gamma = S_\mathcal{A} + S_\mathcal{B} - S_{\mathcal{A}\cup\mathcal{B}}$. In the following section we show, that in the random cluster representation S_Γ is simply given by the mean number of clusters in the optimal sets which are crossed by Γ. This type of problem has already been considered by two of us in the case of the non-random Potts model both for $Q = 1$, representing percolation[28,29] and for general values of $Q \le 4$[30]. Repeating the reasoning applied in these papers we show that the dominant term of S_Γ represents

the area law to which there are logarithmic corrections at the critical point due to corners and these are calculated by conformal techniques. The analytical conformal conjectures are then confronted with the results of large scale numerical calculations for different forms of the contour. These results involve the central charge of the RBPM, for which we calculate a precise estimate.

The rest of the paper is organized as follows. The model, its solution in the random cluster representation and the calculation of the excess entropy is presented in Sec. II. Numerical results for the corner contribution to the excess entropy are presented in Sec. III. Independent estimates for the central charge of the model through analyzing the finite-size correction of the free-energy is given in Sec. IV. Finally, Sec. V contains our conclusions.

II. RANDOM BOND POTTS MODEL IN THE LARGE-Q LIMIT

We consider the Q-state Potts model defined by the Hamiltonian[9]

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\delta(\sigma_i,\sigma_j), \tag{1}$$

in terms of the Potts spin variables $\sigma_i = 0, 1, \ldots, Q-1$. The $J_{ij} > 0$ couplings between nearest neighbor sites are i.i.d. random variables and in the following we restrict ourselves to the square lattice. In the random cluster representation[22] the partition function of the model at $T = 1/\beta$ temperature is given by:

$$\mathcal{Z} = \sum_G Q^{N_{\rm tot}(G)} \prod_{ij\in G} \left[e^{\beta J_{ij}} - 1\right] \tag{2}$$

where the sum runs over all bond configurations G, and in G the total number of connected components (clusters) are denoted by $N_{\rm tot}(G)$. The mean number of clusters is given by:

$$\overline{\langle N_{\rm tot}\rangle} = \frac{\partial \overline{\ln Z(Q)}}{\partial \ln Q}. \tag{3}$$

where $\langle\cdots\rangle$ denotes thermal averaging and $\overline{\cdots}$ stands for the average over quenched disorder.

Let us now introduce a subset of bonds, Γ, and fix all spins on Γ (in state 0, say) but leave the couplings unchanged. Then the partition function becomes:

$$\mathcal{Z}_{\rm fix} = \sum_G Q^{N_{\rm tot}(G)-N_\Gamma(G)} \prod_{ij\in G} \left[e^{\beta J_{ij}} - 1\right] \tag{4}$$

where $N_\Gamma(G)$ denotes the number of clusters which intersect Γ. Consequently:

$$\overline{\langle N_{\rm tot} - N_\Gamma\rangle} = \frac{\partial \overline{\ln Z_{\rm fix}(Q)}}{\partial \ln Q}. \tag{5}$$

Now let us take the large-Q limit, in which case the entropy scales as $S \sim \ln Q$, thus it is convenient to use the reduced entropy: $S' = S/\ln Q$ and the reduced temperature: $T' = T\ln Q$ ($\beta' = \beta/\ln Q$). These reduced quantities are of $\mathcal{O}(1)$ at the phase transition region.

In terms of β' the partition function in Eq.(2) reads as

$$\mathcal{Z} = \sum_G Q^{N_{\rm tot}(G)} \prod_{ij\in G} \left[Q^{\beta' J_{ij}} - 1\right] \tag{6}$$

in which for large-Q we have $Q^{\beta' J_{ij}} \gg 1$, thus

$$\mathcal{Z} = \sum_G Q^{\phi(G)}\ , \quad \phi(G) = N_{\rm tot}(G) + \beta' \sum_{ij\in G} J_{ij}\ , \tag{7}$$

which is dominated by the largest term, $\phi^* = \max_G \phi(G)$. Finally we arrive at

$$\mathcal{Z} = n_0 Q^{\phi^*}\ , \tag{8}$$

where the degeneracy of the optimal set is $n_0 = \mathcal{O}(1)$. The free-energy of the system, F is proportional to the mean value of ϕ^*:

$$\overline{\phi^*} = -\beta' F = S' - \beta' E \tag{9}$$

where the reduced entropy of the system is $S' = \overline{N_{\rm tot}(G^*)}$ and the mean energy is given by: $E = -\overline{\sum_{ij\in G^*} J_{ij}}$. Similarly we obtain for the reduced entropy of the system with a contour of fixed spins as: $S'_{\rm fix} = \overline{N_{\rm tot}(G^*) - N_\Gamma(G^*)}$, consequently the excess entropy associated with the contour is given by the mean number of clusters crossed by Γ:

$$S'_\Gamma = \overline{N_\Gamma}\ . \tag{10}$$

Using this relation S'_Γ can be calculated numerically, which will be performed in the following section.

On the other hand analytical results on S'_Γ can be obtained from the difference of $\overline{\langle N_{\rm tot}\rangle}$ and $\overline{\langle N_{\rm tot} - N_\Gamma\rangle}$ in Eqs.(3) and (5), which is the derivative of $\overline{\ln Z(Q)} - \overline{\ln Z_{\rm fix}(Q)}$. At the critical point this difference is given by: $\sim L_\Gamma f_s(Q) + C_\Gamma \ln L_\Gamma$, where L_Γ is the linear size of the contour, $f_s(Q)$ is the surface free-energy density, which is non-universal and the second term represents the corner contribution[31]. Thus we obtain for the excess entropy:

$$S'_\Gamma = -Q f'_s(Q) L_\Gamma + b_\Gamma(Q)\ln L_\Gamma\ , \tag{11}$$

in which the first term corresponds to the "area-law" and in the second term the prefactor is factorized as:

$$b_\Gamma(Q) = \frac{\partial c(Q)}{\partial \ln Q} A_\Gamma = c' A_\Gamma\ , \tag{12}$$

where $c(Q) = c'\ln Q + \text{cst.}$ is the central charge of the RBPM for large Q. A_Γ is a geometrical factor, which does not depend on Q and it follows from the Cardy-Peschel formula[31]

$$A_\Gamma = \frac{1}{24}\sum_k \left(\frac{\gamma_k}{\pi} - \frac{\pi}{\gamma_k} + \frac{2\pi-\gamma_k}{\pi} - \frac{\pi}{2\pi-\gamma_k}\right)\ , \tag{13}$$

where γ_k is the interior angle at each corner.

III. NUMERICAL RESULTS FOR THE EXCESS ENTROPY

In the numerical calculation we have considered finite samples of size $L \times L$ with periodic boundary conditions and the couplings were taken from a bimodal distribution: $P(\beta' J) = [\delta(w + \Delta w - \beta' J) + \delta(w - \Delta w - \beta' J)]/2$. Having $w = 1/2$ the critical point is given from self-duality[32]: $\beta'_c = 1$. In most of the calculations we have used $\Delta w = 1/3$, but to check universality we have also performed some calculations with $\Delta w = 1/4$. The linear size of the systems were $L = 32, 64, 128, 256$ and 512, and the number of independent samples varied from 80000 at the smallest sizes to more than 1200 for $L = 512$.

The optimal set of a given sample has been calculated by the optimal cooperation algorithm[23], which works in polynomial time and has already been used to solve the RBPM in two-[24,25] and three-dimensional[33] lattices, as well as in scale free networks[34]. If there are multiple optimal sets in the system, then both the intersection and union of any pair of them yields an optimal set. In order to show that our results are independent of the choice of the representing optimal set in a given system, all our studies are carried out for the two limiting cases, namely for the *union* and *intersection* of *all* the optimal sets. We illustrate the cluster structure of the optimal sets in these limiting cases in Fig.1. It is seen, that the intersection consists of a large number of smaller clusters, which are partially merged to common clusters having larger masses in the union. Consequently we have for the averaged number of crossing clusters: $\overline{N}_\Gamma(\text{intersection}) \geq \overline{N}_\Gamma(\text{union})$, however the corner contributions being dominated by the large clusters are expected to be asymptotically identical in the two cases.

Having the optimal sets of different samples we have calculated the excess entropy for different contours: sheared squares, line segments and crosses, which are illustrated in Fig. 2. To subtract the corner contribution from the data we have used the so called geometric approach[28,30,35]: for each sample N_Γ is calculated in two different geometries, which have the same boundary term, but different corner ones. Thus the corner contribution is obtained from their difference. The average in Eq.(10) is performed over i) different samples and ii) over different (~ 1000) positions of the contour in a given sample. For technical details we refer to our previous investigations on the non-random model[28–30].

We start with *sheared squares*, having an opening angle $\gamma \leq \pi/2$ and both its base and altitude is given by $L/2$. In the numerical method we have calculated the corner contribution of the excess entropy for different sizes and then finite-size estimates are calculated for the prefactor b_Γ in Eq.(12) by two point fit. These are presented in Fig.3 for the union and intersection optimal sets. For this contour the geometrical factor, A_Γ in Eq.(13) reads as

$$A_\Gamma = \frac{1}{12}\left[4 - \pi\left(\frac{1}{\gamma} + \frac{1}{\pi - \gamma} + \frac{1}{\pi + \gamma} + \frac{1}{2\pi - \gamma}\right)\right], \tag{14}$$

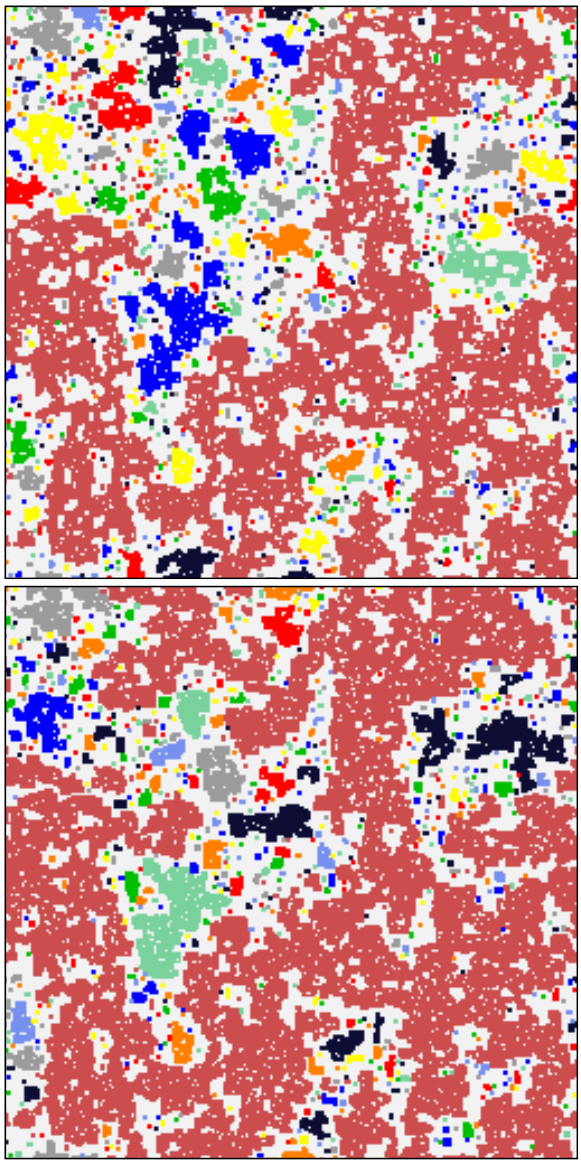

FIG. 1: (Color online) Cluster structure of the same sample in the optimal sets for the intersection (upper panel) and for the union (lower panel) at $L = 256$.

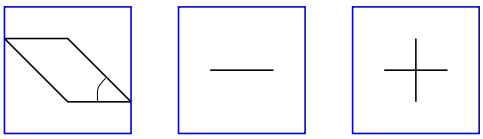

FIG. 2: (Color online) Shape of the subsystems used in the numerical calculations: sheared squares, line segments and crosses.

and we put also the conformal result in Fig.3 with an estimated reduced central charge $c' = 0.74$, see Eq.(16). For not too small opening angles the finite-size estimates are close to the conformal results, for smaller angles the corrections become larger. Extrapolating the prefactors for the two largest angles, $\gamma = \pi/2$ and $\pi/4$ gives approximately the same estimate for the reduced central charge for the intersection and the union, see in Table I.

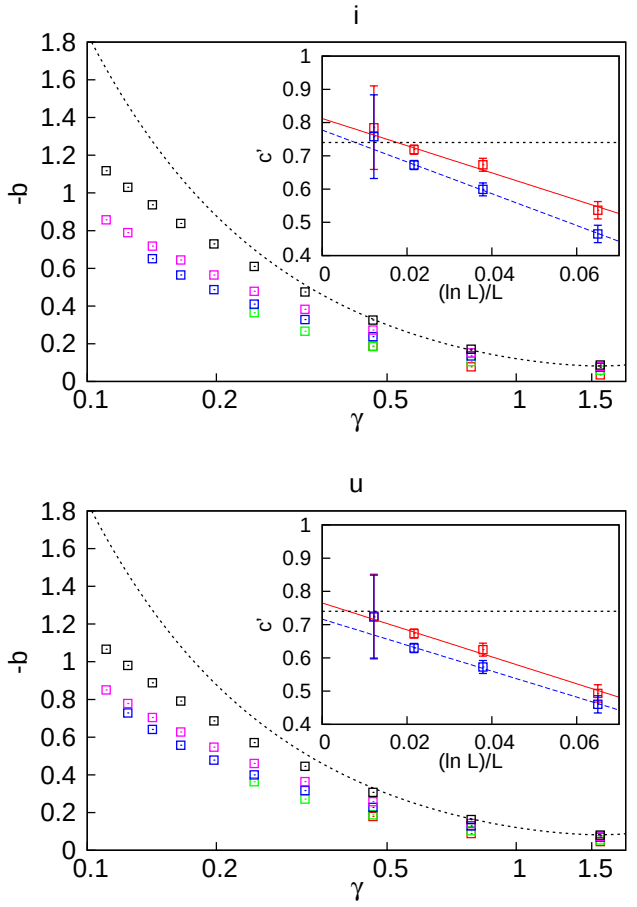

FIG. 3: (Color online) Finite size estimates (L = 32, 64, 128, 256 and 512, from bottom to top) of the prefactor b with sheared squares as a function of γ for intersection (top) and union (bottom) data. For larger sizes the results are approaching the conformal result in Eq.(14), which is indicated by the dotted (black) line. Inset: Finite size estimates of the reduced central charge c' with sheared squares as a function of the system size at $\gamma = \pi/2$ ($\gamma = \pi/4$), indicated by solid (dashed) lines. The estimated value $c' = 0.74$ is shown by the dotted horizontal line.

For *line segments* with length $\ell = L/2$ we have only the corner contributions of two exterior $\gamma = 2\pi$ angles, so that $A_\Gamma = 1/8$. In this case the corner contribution is simply half of the number of common clusters between two line segments. Finite-size estimates for c' are shown in Fig.4 which are extrapolated to the same value (within the accuracy of the calculation) both for intersection and union data, see Table I. We have checked, that the effective central charges for the disorder parameter, $\Delta w = 1/4$, are practically indistinguishable from the results in Fig.4.

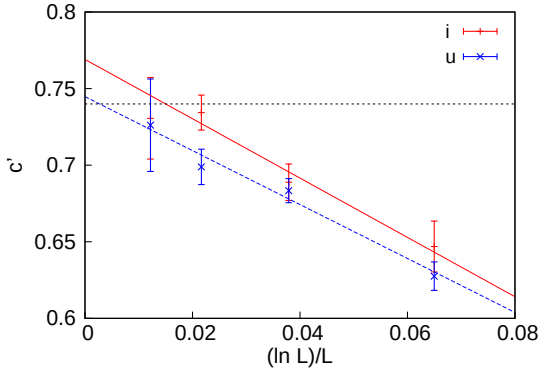

FIG. 4: (Color online) Estimates for the reduced central charge c' for line segments. The estimated value $c' = 0.74$ is indicated by the dotted horizontal line.

For a *cross-like subsystem* the corner contributions are expected to cancel out completely. We have checked, that this is indeed the case, see Fig.5, showing the validity of the Cardy-Peschel formula. We have also studied subsystems comprised of $n = 1, 2, 3$ or 4 crosses (see in Fig.2 of Ref[30]), in which case we have obtained finite-size estimates for c'. For 1 and 3 crosses these are presented in Fig.6 and the extrapolated values in Table I. In this case results for the disorder parameter $\Delta w = 1/4$ leads to the same extrapolated value.

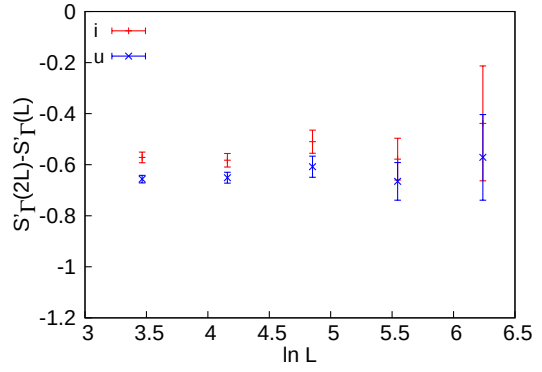

FIG. 5: (Color online) Difference between the excess entropies for one cross with two different sizes: $2L$ and L. There is no size dependence, hence no corner contribution in agreement with the Cardy-Peschel formula.

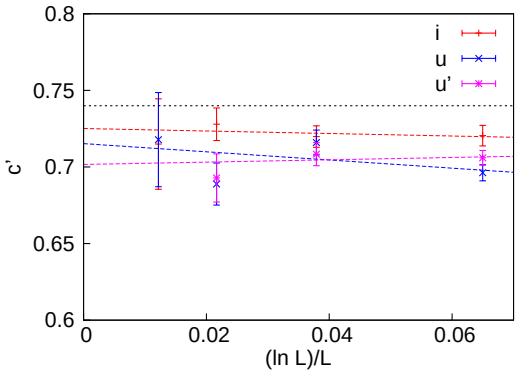

FIG. 6: (Color online) Finite-size estimates for the reduced central charge, c' obtained by comparing the excess entropies for 1 and 3 crosses, for the intersection (i) and the union (u) data. For the union data we present also the results obtained for the disorder parameter $\Delta w = 1/4$ (u'). The estimated value $c' = 0.74$ is indicated by the dotted horizontal line.

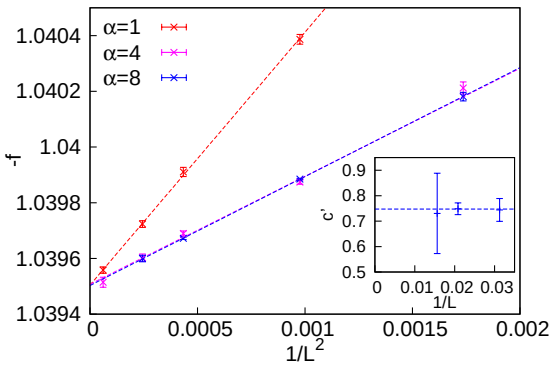

FIG. 7: (Color online) Critical free-energy density of $L \times \alpha L$ systems with periodic boundary conditions as a function of $1/L^2$ for $\alpha = 1$ (square) and for $\alpha = 4$ and $\alpha = 8$ (stripes). The asymptotic value, $f_0 = -1.03951(1)$, is independent of α. The prefactor of the correction term is 0.91(3) for squares and it agrees for both type of stripes. In the inset the effective reduced central charge is shown for $\alpha = 8$ and the dashed horizontal line represents the estimated value in Table I.

IV. CENTRAL CHARGE FROM THE FINITE-SIZE CORRECTION TO THE FREE-ENERGY

The traditional way of subtracting the central charge of a two-dimensional lattice model is to study the finite-size correction to the critical free-energy density in the strip geometry[36,37]. Having the critical RBPM in an infinite strip of width L with periodic boundary conditions the free-energy density scales as:

$$\beta' f(L) = \beta' f_0 + \frac{\pi c'}{6L^2} + \mathcal{O}(L^{-4}) \,, \qquad (15)$$

where $f_0 \equiv f(L = \infty)$ and c' is the reduced central charge in the large-Q limit. Since the free-energy is the same for all optimal sets, we do not make a distinction here between the union and intersection data. In practical adaptation of this method we have used finite stripes of size $L \times \alpha L$ for $L = 24, 32, 48, 64, \ldots$ and at a fixed $\alpha \geq 1$ we have plotted the free-energy densities as a function of $1/L^2$, see Fig. 7. As expected, the limiting value, f_0, does not depend on the aspect ratio, α, but the slopes are different for $\alpha = 1$ and for $\alpha = 4$ and 8. For the latter two values the slopes are very close, therefore we used the data at $\alpha = 8$ to estimate effective finite-size reduced central charge, $c'(L)$ from two-point fits. These are shown in the inset of Fig. 7. As expected the effective $c'(L)$-s have no noticeable size-dependence for $L \geq 32$, since the correction terms are $\mathcal{O}(1/L^2)$. Therefore the extrapolated value given in Table I is simply the mean value of the estimates in the inset of Fig. 7.

TABLE I: Numerical estimates for the reduced central charge c' from the corner contribution of the excess entropy using different contour geometries and from the finite-size correction to the free-energy in stripes.

		union	intersection
excess entropy	squares	0.74(3)	0.79(6)
	lines	0.74(3)	0.77(5)
	crosses	0.72(2)	0.73(1)
free-energy		0.75(2)	

V. CONCLUSION

The random bond Potts model is a basic problem of statistical physics of disordered systems and its large-Q limit is of special interest, when the free energy of the system is dominated by a single diagram, the optimal set. Thermal fluctuations of the $Q \to \infty$ model are negligible compared to disorder fluctuations, therefore its critical properties are controlled by an infinite disorder fixed point. In this paper we considered this model on the square lattice and studied the properties of the excess entropy, S_Γ, associated to a contour of bonds, Γ. S_Γ is found to be proportional to the number of clusters in the optimal set which are crossed by the contour. Using conformal field theory the excess entropy is shown to have a universal corner contribution at the critical point, which scales with the logarithm of the linear size of the contour and its prefactor is proportional to the central charge of the model. We have performed large-scale numerical calculations and confirmed the validity of the conformal prediction. We have also obtained estimates for the central charge of the model, which are collected in Table I for different forms of the contours, as well as for the two extreme forms of the optimal sets. Within the error of

the calculation these all agree with each other, as well as with the results of an independent estimate calculated from the finite-size dependence of the free-energy density. Based on these data we conclude with the estimate:

$$c' = 0.74(2)\ , \tag{16}$$

for the reduced central charge of the model. This value is close to (although somewhat larger than) the previous estimate[20]: $c' = 0.72$ obtained through transfer matrix calculations for large, but finite values of Q. c' in Eq.(16) is expected to be universal for any form of the quenched disorder, for the bimodal distribution it holds for $0 < \Delta < 1/2$. It is different from that at $\Delta = 1/2$, which corresponds to bond percolation having a reduced central charge[28]

$$c'_{\rm perc} = \frac{5\sqrt{3}}{4\pi} \approx 0.689. \tag{17}$$

It remains the subject of other research to study if the central charge of the model is related to some properties of the exactly solved random transverse-field Ising chain[27], as expected from the numerical values of the critical exponents in the two models.

Acknowledgments

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